Asymptotic Optimality for Distributed Spectrum Sharing using Bargaining Solutions

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***Abstract*—Recent studies on spectrum usage reveal poor utilization, both spatially and temporally. Opportunistic use of licensed spectrum while limiting interference to primary users can enhance spectrum reuse and provide orders of magnitude improvement in available channel capacity. This calls for spectrum sharing protocols that are dynamic, flexible, and efficient, in addition to being fair to end users.**

We employ cooperative game theory to address the opportunistic spectrum access problem. Specifically, we develop a game-theoretic model to analyze a scenario in which nodes in a wireless network seek to agree on a fair and efficient allocation of spectrum. First, we show that in high interference environments, the utility space of the game is non-convex, making certain optimal allocations unachievable with pure strategies. To mitigate this, we show that as the number of channels available increases, the utility space approaches convexity, thereby making optimal allocations achievable with pure strategies. Second, by comparing and analyzing three bargaining solutions, we show that the Nash Bargaining Solution achieves the best tradeoff between fairness and efficiency, using a small number of channels. Finally, we develop a distributed algorithm for spectrum sharing that is general enough to accomodate non-zero disagreement points, and show that it achieves allocations reasonably close to the Nash Bargaining Solution.

***Index Terms*—Dynamic spectrum access, cooperation, game theory, radio resource management, Nash bargaining solution.**

I. Introduction

OPPORTUNISTIC spectrum access is a high priority research area, since the limited available spectrum is inefficiently utilized. We consider the problem of nodes in a wireless network that try to gain access to bandwidth by competitively allocating their own transmission power across multiple channels. We specifically study the case where no coordinating entity exists in the network, and nodes seek to arrive at a fair and efficient sharing of available channels in a distributed fashion. Because only local information is available and control signals impose much overhead in environments such as multihop wireless networks, distributed protocols are preferred, which motivates us to employ game theory in the analysis and design of spectrum sharing mechanisms.

Game theory is an analytical tool often used for modeling interactive decision making. Game-theoretic models are particularly suitable in analyzing distributed resource management problems where user performance may be interdependent, as these users compete for resources. Game theory provides a rich and flexible framework for analyzing dynamic systems and predicting their outcomes as users optimize their performance locally, possibly in the face of limited network state information. Game theory is often classified into two categories: cooperative and non-cooperative. One of the key problems addressed by cooperative game theory is how to split or allocate resources among users in a coalition. Cooperative game theory also strives to understand the interplay between an efficient and a fair allocation. The problem of spectrum sharing among secondary users can be mapped into a cooperative game. System efficiency and fairness are vital in any spectrum allocation problem, yet these objectives are often at odds with each other. Systems optimized for maximizing spectrum utilization often suffer from starvation issues, where users are allocated resources disproportionately. Using cooperative game theory, we align these two desirable objectives.

The focus of this work is on achieving the Nash Bargaining Solution (NBS) allocation of power, and evaluating it with respect to efficiency and fairness. Our primary contribution is to analyze the utility space of the spectrum sharing game and show that the NBS achieves a fair and efficient spectrum allocation. We show that in a high interference environment with a finite number of channels, the utility space of the spectrum sharing game is non-convex. Non-convexity can lead to optimal operating points that require mixed strategies.[1] We show that by increasing the number of channels available, while keeping the number of users fixed, the utility space becomes closer to convex and more optimal operating points can be achieved with pure strategies.

Another contribution of our work is to show that the NBS allocation provides a reasonable compromise between efficiency and fairness. We provide a discussion of cooperative game theory, where we introduce three of the most popular bargaining solutions, the egalitarian, Kalai-Smorodinsky and Nash bargaining solutions, and show how they can be applied to non-convex utility spaces. In our simulation results, we show that the NBS provides the best compromise between fairness and efficiency even with a small number of channels.

Finally, we propose an algorithm that can achieve the

Manuscript received October 11, 2008; revised May 2, 2009; accepted June 14, 2009. The associate editor coordinating the review of this paper and approving it for publication was E. Hossain.

This material is based on work supported in part by the Lynde and Harry Bradley Graduate Fellowship, and by the National Science Foundation under Grant No. 0448131.

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Digital Object Identifier 10.1109/TWC.2009.081340

[1] A mixed strategy is composed of a set of possible actions and a probability distribution over this action space. A pure strategy is a mixed strategy that consists of a single possible action.

TABLE I
A LIST OF NOTATIONS AND ABBREVIATIONS USED IN THIS PAPER.

Notation	Meaning	Abbr.	Meaning
N	Number of transmitters (players)	NBS	Nash bargaining solution
U	Utility space	ES	Egalitarian solution
$\mathbf{u}$	Utility vector in U	KSS	Kalai-Smorodinsky solution
$\mathbf{u^0}$	Utility at disagreement point	PO	Pareto optimal
$\phi(U, \mathbf{u^0})$	Bargaining solution	WPO	Weak Pareto optimal
$h(U, \mathbf{u^0})$	Utopia point vector	NP	Nash Product
U_c	Convex hull of U	IIA	Independence of irrelevant alternatives
χ	Set of channels	INV	Invariance to affine transformations
K	Number of channels	RMON	Restricted monotonicity
k	Channel index; an element of χ	SMON	Strong monotonicity
B	Aggregate bandwidth	IZ	Interference zone
p_i	Transmit power of node i	IR	Individual rationality
$P_{\max}$	Max power constraint for nodes	MaxSum	Allocation maximizing sum of capacities
p_i^k	Power level of node i on channel k		
P_i^χ	Set of power levels of node i over all channels		
R	Interference radius		
R_a	Dimensions of the square region that contains the nodes		
δ	Search space controlling parameter		

NBS allocation in a distributed manner, using only local information. The algorithm focuses not on implementing a bargaining process between players, but on achieving the NBS, since that is the expected outcome of the bargaining process. Finding the NBS point requires solving a non-convex optimization problem, and our algorithm implements a distributed approximation to the solution of this problem. We assume the network is stable enough that the information gathered by nodes does not significantly change during the execution of the algorithm. In our simulation results, we show that the algorithm reasonably approximates the NBS allocation using only information from neighboring nodes.

The work presented here represents the continuation of the work published in [1]. We extend that work by presenting an analysis of three different bargaining solutions and how they are affected by non-convex utility spaces. Additionally, the algorithm presented here takes advantage of a logarithmic transformation of the Nash product so it can handle non-zero disagreement points. Finally, we conduct more detailed simulation results to compare the NBS with other bargaining solutions.

The remainder of the paper is organized as follows. Section II provides a brief description of cooperative game theory and the bargaining solutions, which are required to understand the development of results in the paper. Section III discusses the bargaining solutions for non-convex sets. Section IV analyzes the spectrum sharing problem, including the game model, the utility space, and the NBS in the context of the spectrum sharing game. Section V presents the distributed algorithm and shows that it converges. Section VI provides our simulation results. Section VII discusses the literature most relevant to our work. Finally, Section VIII provides some concluding remarks.

II. Cooperative Game Theory and Nash Bargaining

Game theory provides a set of mathematical tools that are useful in analyzing complex decision problems with interactions among self-interested decision makers, called players. The basic component of game theory is a game, $G = \langle M, A, \{u_i\}\rangle$. $M = \{1, \ldots N\}$ is the set of players, A_i is the set of actions available to player i, $A = A_1 \times ... \times A_N$, and u_i is the objective function, sometimes called the utility function, that player i wishes to maximize. A vector of such utilities is denoted by $\mathbf{u} = (u_1, \ldots, u_N)$; define $U = \{u(\mathbf{a}) \mid \mathbf{a} \in A\}$ as the set of achievable utility for all players.

In a non-cooperative game, players individually attempt to maximize their own utility without regard to the utility achieved by other players. In a cooperative game, players bargain with each other. If an agreement is reached, players act according to the agreement reached. If the players cannot reach an agreement, players act in a non-cooperative way. Note that the agreements reached must be binding, so players are not allowed to deviate from what is agreed upon. For instance, in the context of our model, a regulator (in the US, the FCC) may take the role of authority that punishes individuals (by fines, license suspension, etc.) for not behaving according to pre-established policies for spectrum sharing.

John Nash wrote in his seminal paper on cooperative games [2] that to understand the outcome of a bargaining game, we should not focus on trying to model the bargaining process itself, but instead, we should list the properties, or axioms, that we expect the outcome of the bargaining process to exhibit. Once we define these axioms, we can eliminate all possible outcomes that do not satisfy the axioms. This narrows down the possible outcomes of the bargaining process, but we cannot distinguish which of all possible remaining outcomes is likely to occur. So, if we define the axioms such that only one possible outcome satisfies them, there is no ambiguity in the outcome of the bargaining process. This way of analyzing cooperative games is called *axiomatic bargaining game theory* [3]. Before we proceed, we need to introduce some terminology and notations.

1) *Agreement point* - an action vector $\mathbf{a} \in A$ that is a possible outcome of the bargaining process.
2) *Disagreement point* - an action vector $\mathbf{a} \in A$ that is expected to be the result of non-cooperative play given a failure of the bargaining process. Clearly, the utility achieved by every player at any agreement point has to be at least as much as the utility achieved at the disagreement point.
3) *Pareto optimal* - $\mathbf{u} \in U$ is Pareto optimal if $\nexists \mathbf{u'} \in U$ such that $u_i^{'} \geq u_i\ \forall i$ and $u_i^{'} > u_i$ for some i. When

the inequality is strict, $\mathbf{u}$ is *weak* Pareto optimal. Define $\texttt{PO}(U)$ as the set of Pareto optimal points of U and $\texttt{WPO}(U)$ as the set of weak Pareto optimal points of U. Notice that $\texttt{PO}(U) \subseteq \texttt{WPO}(U)$.

4) *Bargaining problem* - a vector containing a utility space and a disagreement point for a game.
5) *Bargaining solution* - A bargaining solution ϕ, defined on a class of bargaining problems Σ, is a map that associates with each problem $(U, \mathbf{u}^0) \in \Sigma$ a unique point in U, where $\mathbf{u}^0 = \mathbf{u}(\mathbf{a}^0)$ is the utility achieved at the disagreement point $\mathbf{a}^0$.

To determine the bargaining solutions, the following conditions about the utility space U apply:

- $U \subset \mathbb{R}^N$ is upper-bounded, closed and convex.
- There exists $\mathbf{u} \in U$ such that $\mathbf{u}^0 < \mathbf{u}$.

Nash proposed the following axioms:

1) Individual rationality (IR): $\phi(U, \mathbf{u}^0) > \mathbf{u}^0$.
2) Pareto optimality (PO): $\phi(U, \mathbf{u}^0) \in \texttt{PO}(U)$.
3) Invariance to affine transformations (INV): if $\psi : \mathbb{R}^N \to \mathbb{R}^N$, $\psi(\mathbf{u}) = \mathbf{u}'$ with $u'_i = c_i u_i + d_i, c_i, d_i \in \mathbb{R}, c_i > 0, \forall i$, then $\phi(\psi(U), \psi(\mathbf{u}^0)) = \psi(\phi(U, \mathbf{u}^0))$.
4) Independence of irrelevant alternatives (IIA): if $\mathbf{u}' \in V \subset U$ and $\mathbf{u}' = \phi(U, \mathbf{u}^0)$ then $\phi(V, \mathbf{u}^0) = \mathbf{u}'$.
5) Symmetry (SYM): if U is symmetric with respect to i and j, $u_i^0 = u_j^0$, and $\mathbf{u}' = \phi(U, \mathbf{u}^0)$, then $u'_i = u'_j$.

The INV axiom assures that the solution is invariant if affinely scaled. The IIA axiom states that if the domain is reduced to a subset of the domain that contains the bargaining solution, then the solution remains the same. The SYM axiom states that the solution does not depend on the labels, i.e. if two players have the same disagreement utility and the same set of feasible utility, then they will achieve the same utility at the solution.

The following definition of NBS was first proposed by Nash for two-player games [2], and later extended for more than two players [3].

Definition 1: **Nash Bargaining Solution**. $\phi(U, \mathbf{u}^0)$ is the NBS if and only if it satisfies the axioms IR, PO, INV, SYM and IIA.

When U is convex, Nash showed that the unique NBS is the maximizer of the Nash Product (NP),

$$\arg\max_{\mathbf{u} > \mathbf{u}^0} \prod_{i=1}^{N} (u_i - u_i^0). \tag{1}$$

Bargaining solutions that have been proposed as alternatives to the NBS include the Kalai-Smorodinsky solution (KSS) [4] and the Egalitarian solution (ES). To define these solutions, we need to introduce the following axioms:

1) Strong monotonicity (SMON) - if $V \subset U$ then $\phi(U, \mathbf{u}^0) \geq \phi(V, \mathbf{u}^0)$.
2) Restricted monotonicity (RMON) - if $V \subset U$ and $h(U, \mathbf{u}^0) = h(V, \mathbf{u}^0)$ then $\phi(U, \mathbf{u}^0) \geq \phi(V, \mathbf{u}^0)$, where $h(U, \mathbf{u}^0)$, called the *utopia point*, is defined as:
$$h(U, \mathbf{u}^0) = (\max_{\mathbf{u} > \mathbf{u}^0} u_1(\mathbf{u}), \ldots, \max_{\mathbf{u} > \mathbf{u}^0} u_N(\mathbf{u})). \tag{2}$$
3) Weak Pareto optimality (WPO) - $\phi(U, \mathbf{u}^0) \in \texttt{WPO}(U)$.

Definition 2: **Kalai-Smorodinsky Solution**. $\phi(U, \mathbf{u}^0)$ is the KSS if and only if it satisfies the axioms IR, PO, INV, SYM and RMON.

Let Λ be set of points in the line containing $\mathbf{u}^0$ and $h(U, \mathbf{u}^0)$. The unique KSS is $\texttt{WPO}(U) \cap \Lambda$, which can be expressed as[2]:

$$\max \left\{ \mathbf{u} > \mathbf{u}^0 | \frac{1}{\theta_i}(u_i - u_i^0) = \frac{1}{\theta_j}(u_j - u_j^0), \forall i, j \right\} \tag{3}$$

where $\theta_i = h_i(U, \mathbf{u}^0) - u_i^0$.

Definition 3: **Egalitarian Solution**. $\phi(U, \mathbf{u}^0)$ is the ES if and only if it satisfies the axioms IR, WPO, SYM and SMON. Let $\mathbf{u}'$ be such that $\forall i,\ u'_i = u_i^0 + b, b \in \mathbb{R}$ and Λ be set of points in the line containing $\mathbf{u}^0$ and $\mathbf{u}'$. The unique ES is $\texttt{WPO}(U) \cap \Lambda$, which can be expressed as:

$$\max \left\{ \mathbf{u} > \mathbf{u}^0 | u_i - u_i^0 = u_j - u_j^0, \forall i, j \right\}. \tag{4}$$

The KSS assigns as the bargaining solution the point in the Pareto set that intersects the line connecting the disagreement point and the utopia point. The ES assigns as the bargaining solution the point in the weak Pareto set where all players achieve equal increase in utility relative to the disagreement point.

III. Bargaining Solutions for Non-convex Sets

In this section we discuss how the bargaining solutions just discussed also apply to a special class of non-convex utility spaces under certain conditions. Specifically, we assume the following conditions:

1) $U \subset \mathbb{R}^N$ is upper-bounded, closed and $\mathbf{u}^0$-comprehensive. U is said to be $\mathbf{u}^0$-comprehensive if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ such that $\mathbf{u}^0 \leq \mathbf{y} \leq \mathbf{x}$, then $\mathbf{x} \in U$ implies $\mathbf{y} \in U$.
2) There exists $\mathbf{u} \in U$ such that $\mathbf{u}^0 < \mathbf{u}$.

Let U_c be the smallest convex set such that $U \subset U_c$; U_c is called the convex hull of U. In [5] it is shown that for such utility spaces the NP maximizer only satisfies the INV, IR and IIA axioms. Noteworthy is that [5] shows that the PO axiom is not guaranteed to be satisfied. This is because the author considers PO in the context of all possible mixed strategies (which we may think of as convex combinations of pure strategies). So, it is possible for a mixed strategy to obtain an expected utility that Pareto dominates the NP maximizer of U. In the remainder of this manuscript, we refer to Pareto optimality with respect to only pure strategies. So, in this sense, the NP maximizer of U is PO. We state the following theorem which describes the conditions under which the NP maximizer for U is the unique NBS (and thus satisfies all NBS axioms).

Theorem 1: If U has a unique NP maximizer, $\mathbf{u}^*$, which coincides with the NP maximizer of U_c, $\mathbf{u}_c^*$, then $\mathbf{u}^*$ is the unique NBS for U.

Proof: We know that $U \subset U_c$. Since the unique maximizer of the NP for both U and U_c are the same, this implies $u^* = u_c^* \in U$. Thus, by the IIA axiom, u^* is the unique NBS for U. ■

In [6] it is shown that the KSS and the ES satisfy the same axioms as for convex sets, with the exception that the KSS satisfies the WPO axiom, instead of the PO axiom. Since PO is a desirable property, we develop extensions of these solutions to include PO.

[2] $\texttt{max}$ with respect to the partial order of $\mathbb{R}^N$

Definition 4: **PO Extension**. Let $\mathbf{u} \in \mathtt{WPO}(U)$ be a bargaining solution, then $\mathbf{u}'$ is a PO extension of $\mathbf{u}$ if $\mathbf{u}' \geq \mathbf{u}$ and $\mathbf{u}' \in \mathtt{PO}(U)$.

Theorem 2: Define the following set:

$$\mathtt{PO}(U) \cap \arg\max_{\mathbf{u}>\mathbf{u}^0} \min_i \{\frac{1}{\theta_i}(u_i - u_i^0)\}. \quad (5)$$

When $\theta_i = h_i(U, \mathbf{u}^0) - u_i^0$, all points in the set are PO extensions of the KSS and when $\theta_i = 1$ all points are PO extensions of the ES.

Proof: First, we show that the set defined in (5) is not empty. Let $\theta_i > 0$,

$$MM = \arg\max_{\mathbf{u}>\mathbf{u}^0} \min_i \{\frac{1}{\theta_i}(u_i - u_i^0)\} \quad (6)$$

and $\mathbf{v} \in MM$. By definition, there does not exist $\mathbf{v}' \in U$ such that $\mathbf{v} < \mathbf{v}'$, which implies $\mathbf{v} \in \mathtt{WPO(U)}$. If $\mathbf{v} \notin \mathtt{PO(U)}$, then because U is upper-bounded, there must exist $\mathbf{v}' \in U$ such that $\mathbf{v} \leq \mathbf{v}'$ and $\mathbf{v}' \in \mathtt{PO(U)}$. Since $\mathbf{v} \in MM$, it must be that $\mathbf{v}' \in MM$.

Let $\mathbf{u}$ be the defined as (3), and $\mathbf{v}$ be in the set defined in (5). Assume that $\mathbf{u} \nleq \mathbf{v}$, which implies there exists j such that $\frac{1}{\theta_j}(u_j - u_j^0) > \frac{1}{\theta_j}(v_j - v_j^0)$. However, we know that $\min_i \frac{1}{\theta_i}(u_i - u_i^0) = \frac{1}{\theta_j}(u_j - u_j^0)$. This contradicts the fact that $\min_i \frac{1}{\theta_j}(v_j - v_j^0) \geq \min_i \frac{1}{\theta_i}(u_i - u_i^0)$ for all $\mathbf{u} > \mathbf{u}^0$. Thus, $\mathbf{u} \leq \mathbf{v}$. ■

We examine the example utility space in Figure 2 (the disagreement point is the origin). Point A is the NP maximizer for both U and U_c, so it is the unique NBS for both. Point B indicates the KSS, which is WPO. The only PO extension of the KSS is point A. Similarly, point C is the ES and the PO extension is point D. Notice that the PO extensions of the KSS and the NBS coincide.

We will subsequently show that the utility space for the spectrum sharing problem is $\mathbf{u}^0$-comprehensive and not always convex, but given the appropriate conditions, can approach convexity. When nearly convex, as in the example, the NP maximizer is the unique NBS and it is often a PO extension of the KSS.

IV. Spectrum Sharing

The spectrum sharing problem addresses the issue of how to allocate the limited available spectrum among multiple wireless devices. The problem has two important, sometimes orthogonal goals: efficiency and fairness. The allocation of spectrum should utilize the resource as efficiently as possible. However, when efficiency is maximized, fairness can be compromised.

We propose a game model and discuss our assumptions. We then analyze the resulting utility space of the game, examine its properties and show that when the available spectrum is divided into a large enough number of channels, efficient spectrum allocation can be achieved with a pure strategy. Finally, we show that the utility space for the spectrum sharing game meets the criteria for non-convex sets discussed in section III.

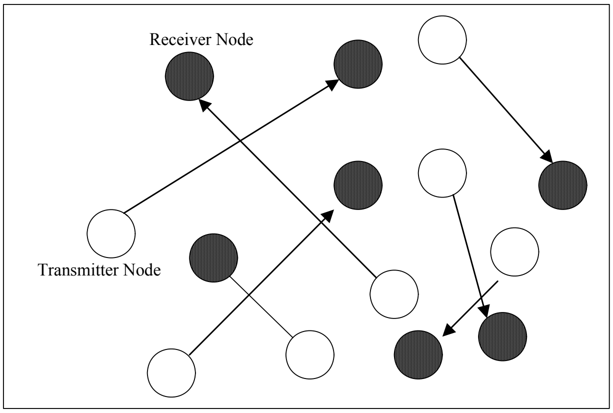


Fig. 1. An instantaneous snapshot of an ad hoc network with possibly multiple interfering transmissions on multiple channels.

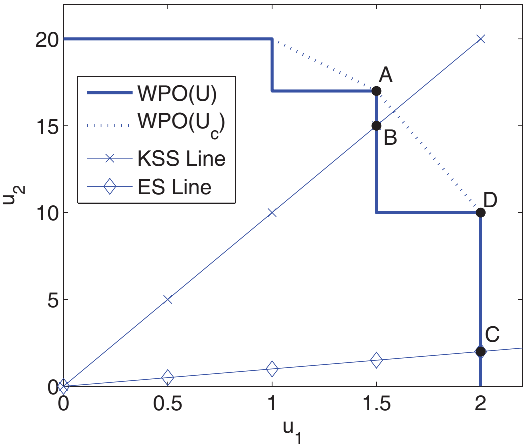


Fig. 2. Example non-convex utility spaces for a two player game. Disagreement point is the origin. A is the NP maximizer and a PO extension of KSS, B indicates the KSS, C is the ES, and D is the PO extension.

A. System Model

The spectrum sharing problem can be modeled as follows. The available bandwidth is divided equally into multiple orthogonal channels. Each wireless device (referred to as a node) can transmit in any combination of channels at any time and can set its transmit power on each channel subject to its maximum power constraint (on total power). Allocating power this way makes sense in multiuser orthogonal frequency division multiplexing (OFDM) systems, even if users are transmitting on contiguous channels. By allocating different powers (analogous to multiuser water filling) across different channels, it can also be argued that users can achieve at least as high a data rate as that achieved by allocating power across a wider band (by combining a block of consecutive channels). We model a snapshot of the network in time, which may be viewed as one in which each transmitting node is communicating to a single receiver node as illustrated in Fig. 1. In other words, at any instance in time during the course of a game, the network appears as a collection of transmitter-receiver pairs. Receiver nodes do not transmit and thus are not considered players in the game (since they will act in coordination with the transmitters).

Let $\chi = \{1, \ldots, K\}$ be the set of available channels, B be the aggregate bandwidth, with each channel in χ having bandwidth $\frac{B}{K}$, and N be the number of transmitter nodes in the network.[3] Each node i allocates power p_i^k on channel $k \in \chi$ subject to its maximum transmit power constraint P_{max}. We can then formulate the spectrum sharing game as follows: $M = \{1, \ldots, N\}$ is the player set, $P_i^\chi = \{(p_i^k)_{k\in\chi} \mid p_i^k \geq 0, \sum_{k\in\chi} p_i^k \leq P_{max}\}$ is the set of all power levels for user i, and $P^\chi = P_1^\chi \times \cdots \times P_N^\chi$ is the space of all power vectors. For a given power allocation vector $\mathbf{p} \in P^\chi$, user i's feasible aggregate rate (over all subcarriers) is given by the Shannon capacity formula:

$$u_i(\mathbf{p}) = \frac{B}{K}\sum_{k=1}^{K}\log_2\left(1 + \frac{H_{ii}^k p_i^k}{\frac{\sigma^2}{K} + \sum_{j\neq i} H_{ji}^k p_j^k}\right) \quad (7)$$

where H_{ji}^k is the channel gain from j to the receiver of i on channel k, and σ^2 is the thermal noise for the entire bandwidth B. We assume that channel gains and interference powers are known a priori; this information may be obtained by means of pilot transmissions and feedback mechanisms. The i-th user's data rate for a specific channel k (from expression (7)) corresponds to the capacity of an additive white gaussian noise (AWGN) channel for a given received signal to interference and noise ratio (SINR). We treat all interference from other users as AWGN. Note that we ignore the cost of spectrum access (e.g. scanning and switching channels), which is beyond the scope of this paper. Note also that we are implicitly modeling systems that do not use techniques such as successive interference cancellation within a subcarrier. Such techniques are generally too complex for practical systems.

B. Utility Space

This section discusses the properties of the utility space for the spectrum sharing game. Specifically, we explore the effect of increasing the number of channels on the utility space. We show that, given enough channels, for any mixed strategy we can find a pure strategy that achieves a utility at least as high as the mixed strategy. This result implies that to achieve efficient spectrum use we need not employ mixed strategies. This also implies that in the cases where the utility space is not convex (some mixed strategies are not included in the space), increasing the number of channels increases the number of mixed strategies that are included in the set (i.e., the utility space approaches convexity). Note that sharing channels through time division can achieve convexity of the utility space. However such a time-sharing mechanism often requires precise synchronization of time slots, which is difficult to achieve; in addition, time-sharing limits spatial reuse.

Theorem 3: For some χ and finite subset $S^\chi \subset P^\chi$, consider a mixed strategy defined by probability distribution π, such that $\pi(\mathbf{s})$ is rational for all $\mathbf{s} \in S^\chi$. For any such mixed strategy, there exists a pure strategy $\mathbf{s} \in P^{\chi'}$, with $P^{\chi'}$ associated with a set of channels χ', that yields the same utility as the mixed strategy.

[3]Note that achieving channel division and determining the number of channels needed is similar to the problem to determining the number of subcarriers in an OFDM system (which is based on achieving flat fading characteristics).

Proof: By construction, for any $\mathbf{s} \in S^\chi$ and mixed strategy π, there exists positive numbers $a_\mathbf{s}, \beta$ such that $\pi(\mathbf{s}) = \frac{a_\mathbf{s}}{\beta}$ ($\sum_{\mathbf{s}\in S^\chi} a_\mathbf{s} = \beta$). We will now find a pure strategy that yields the same utility as this pure strategy. Let $\chi' = \{1, \ldots, \beta K\}$. Let us partition χ' into disjoint sets $\phi_{\mathbf{s},k}$, $\mathbf{s} \in S^\chi$ and $k \in \chi$, with $\bigcup_{\mathbf{s}\in S^\chi, k\in\chi} \phi_{\mathbf{s},k} = \chi'$ and with $|\phi_{\mathbf{s},k}| = a_\mathbf{s}$. It is assumed that for all $k' \in \phi_{\mathbf{s},k}, H_{ij}^{k'} = H_{ij}^k$ for all $i, j \in M$. Let us construct a pure strategy, $\mathbf{t} \in P^{\chi'}$, as follows: for each $k' \in \chi'$, find $\mathbf{s}, k$ such that $k' \in \phi_{\mathbf{s},k}$ and for each $i \in M$ set $t_i^{k'} = \frac{s_i^k}{\beta}$.

Note that t is a valid strategy, as:

$$\sum_{k'=1}^{\beta K} t_i^{k'} = \sum_{\mathbf{s}\in S^\chi}\sum_{k=1}^{K}\sum_{k'\in\phi_{\mathbf{s},k}} t_i^{k'} = \sum_{\mathbf{s}\in S^\chi}\pi(\mathbf{s})\sum_{k=1}^{K} s_i^k \leq P_{max}. \quad (8)$$

The utility achieved by player i for strategy $\mathbf{t}$, $u_i(\mathbf{t})$, is,

$$\begin{aligned}
&\sum_{\mathbf{s}\in S^\chi}\sum_{k=1}^{K}\sum_{k'\in\phi_{\mathbf{s},k}} \frac{B}{\beta K}\log_2\left(1 + \frac{H_{ii}^{k'} t_i^{k'}}{\frac{\sigma^2}{\beta K} + \sum_{l\neq i} H_{li}^{k'} t_l^{k'}}\right)\\
=\ &\sum_{\mathbf{s}\in S^\chi}\sum_{k=1}^{K}\sum_{k'\in\phi_{\mathbf{s},k}} \frac{B}{\beta K}\log_2\left(1 + \frac{H_{ii}^{k} s_i^{k}}{\frac{\sigma^2}{K} + \sum_{l\neq i} H_{li}^{k} s_l^{k}}\right)\\
=\ &\sum_{\mathbf{s}\in S^\chi}\frac{a_\mathbf{s}}{\beta}\sum_{k=1}^{K}\frac{B}{K}\log_2\left(1 + \frac{H_{ii}^{k} s_i^{k}}{\frac{\sigma^2}{K} + \sum_{l\neq i} H_{li}^{k} s_l^{k}}\right)\\
=\ &\sum_{\mathbf{s}\in S^\chi}\pi(\mathbf{s})u_i(\mathbf{s}) = E_\pi[u_i(\mathbf{s})]. \quad (9)
\end{aligned}$$

So, the pure strategy $\mathbf{t}$ achieves a utility equal to the expected utility of the mixed strategy $\mathbf{s}$. ■

Theorem 3 says that we can replicate any mixed strategy with a pure strategy[4].

This result implies that, given enough channels, we do not need to employ mixed strategies to achieve efficient spectrum utilization. Another less evident, yet important, implication is that we can make the utility space closer to convex by increasing the number of channels. If we can replicate mixed strategies with pure strategies, we reduce the number of convex combinations not included in the utility space and thus make it closer to convex. In the remainder of this section we investigate the behavior of the utility space as the number of channels increases. For purposes of illustration, we consider a game with only two players.

Figure 3 shows the utility space for a case where both nodes experience interference stronger than their received signal strength ($H_{21}^k \gg H_{11}^k, H_{12}^k \gg H_{22}^k, \forall k$), for increasing values of K. The figure also shows the utility space when all possible mixed strategies (the convex hull of the utility space) are included for some values of K.

By examining the utility space for $K = 1$, we can clearly see that the mixed strategies dominate the pure strategies. This shows that under the high interference case for single channel, using mixed strategies is required for fairness and efficiency. Now we examine the utility space for $K = 4$. We notice that the WPO set for this case dominates most of the WPO set of the $K = 1$ case with the mixed strategies. Specifically, the

[4]We can only replicate mixed strategies with rational probabilities, but since rationals are dense in the reals, we can approximate any mixed strategy arbitrarily closely.

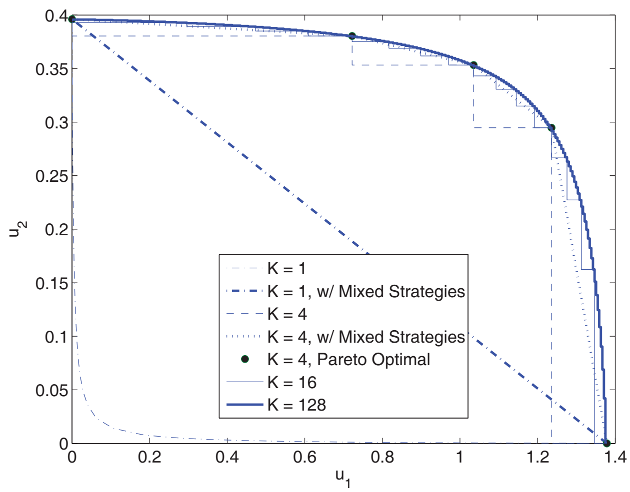


Fig. 3. WPO set for 2 player game under high interference for various values of K.

PO points for $K = 4$ overwhelmingly dominate the mixed strategies for $K = 1$. Also, these points are not dominated by the mixed strategies for the case $K = 4$.

From the results of Theorem 3, we would expect the boundary for $K = 4$ to be at least as efficient as some mixed strategies for the $K = 1$ case. The overwhelming dominance of the $K = 4$ case over the $K = 1$ case is due also in part to an effect that is most noticeable in high interference environments. When interference is high, optimal mixed strategies usually involve only a single node transmitting at a time. By increasing the number of channels, we allow for frequency separation of transmissions and thus decrease interference. The decrease in available bandwidth to each player is more than offset by the decrease in interference. Corollary 1 formally presents this effect.

Corollary 1: Consider a mixed strategy as in Theorem 3, such that $s_i^k > 0$ for some $k \in \chi$ implies $s_j^k = 0$ for all $j \neq i$ and $k \in \chi$. Also, for every i there exists $\mathbf{s} \in S^\chi$ such that $s_i^k > 0$ for some k. For any such mixed strategy, there exists a pure strategy $\mathbf{t} \in P^{\chi'}$, with $P^{\chi'}$ associated with a set of channels χ', that yields a utility greater than that of the mixed strategy.

Proof: Let S_i^χ be the set of strategies such that for some $k \in \chi$, $s_i^k > 0$. By construction, $|S_i^\chi| > 0$ and $\bigcup_{i \in M} S_i^\chi = S^\chi$, and thus S_i^χ is a proper subset of S^χ. Now consider the pure strategy $\mathbf{t} \in P^{\chi'}$ as defined in the proof for theorem 3. Let $\beta_i = \sum_{\mathbf{s} \in S_i^\chi} a_\mathbf{s} < \beta$ and let us construct another pure strategy, $\mathbf{v} \in P^{\chi'}$, such that $v_i^{k'} = \frac{t_i^{k'}\beta}{\beta_i} > t_i^{k'}$.

Note that $\mathbf{v}$ is a valid strategy, as:

$$\sum_{k'=1}^{\beta K} v_i^{k'} = \sum_{\mathbf{s} \in S_i^\chi} \sum_{k=1}^{K} \sum_{k' \in \phi_{\mathbf{s},k}} v_i^{k'} = \sum_{\mathbf{s} \in S_i^\chi} \frac{a_\mathbf{s}}{\beta_i} \sum_{k=1}^{K} t_i^k \leq P_{max}. \tag{10}$$

The utility achieved by player i for strategy $\mathbf{v}$, $u_i(\mathbf{v})$, is,

$$\begin{aligned} &\sum_{\mathbf{s} \in S^\chi} \sum_{k=1}^{K} \sum_{k' \in \phi_{\mathbf{s},k}} \frac{B}{\beta K} \log_2 \left(1 + \frac{H_{ii}^{k'} v_i^{k'}}{\frac{\sigma^2}{\beta K} + \sum_{l \neq i} H_{li}^{k'} v_l^{k'}} \right) \\ &= \sum_{\mathbf{s} \in S^\chi} \sum_{k=1}^{K} \sum_{k' \in \phi_{\mathbf{s},k}} \frac{B}{\beta K} \log_2 \left(1 + \frac{H_{ii}^k \frac{s_i^k \beta}{\beta_i}}{\frac{\sigma^2}{K}} \right) \\ &> \sum_{\mathbf{s} \in S^\chi} \frac{a_\mathbf{s}}{\beta} \sum_{k=1}^{K} \frac{B}{K} \log_2 \left(1 + \frac{H_{ii}^k s_i^k}{\frac{\sigma^2}{K}} \right) = E_\pi[u_i(\mathbf{s})]. \end{aligned} \tag{11}$$

So, the pure strategy $\mathbf{v}$ achieves a utility greater than the expected utility of the mixed strategy $\mathbf{s}$. ■

Now compare the WPO set for $K = 4$ with and without mixed strategies. We can see that using the mixed strategies can achieve the PO points that are not achievable with pure strategies. Although these points do not dominate the pure strategy PO points, it may be desirable to achieve them to meet the fairness objectives. Finally, we examine the utility spaces for $K > 4$. We can see that as K increases, the pure strategy utility space has more PO points. A more subtle, yet important effect of increasing channels is that the utility space approaches its convex hull. As the utility space approaches convexity, we can achieve more PO points with the pure strategies.

C. Bargaining Solutions

We wish to analyze the spectrum sharing game using cooperative game theory. In the previous section we have shown that the utility space for the spectrum sharing game is not always convex. The following theorem shows that the utility space for the spectrum sharing game is $\mathbf{u}^0$-comprehensive, which allows us to use the results of section III.

Theorem 4: The utility space for the spectrum sharing game is $\mathbf{u}^0$-comprehensive.

Proof: Let $\mathbf{u}(\cdot) = (u_1(\cdot), \ldots, u_N(\cdot))$ and $\mathbf{p} \in P^\chi$ be such that $\mathbf{u}(\mathbf{p}) = \mathbf{x} \geq \mathbf{y} \geq \mathbf{u}^0$. Let $1 \leq j \leq N$, $0 \leq \alpha \leq 1$, and define $\hat{\mathbf{p}}$ such that $\hat{p}_i^k = p_i^k$ for all $i \neq j$, and $\hat{p}_j^k = \alpha p_j^k$. We know $u_j(\hat{\mathbf{p}}) = 0$ for $\alpha = 0$ and $u_j(\hat{\mathbf{p}}) = x_j$ for $\alpha = 1$. Then, the intermediate value theorem guarantees that there exists $0 \leq \alpha \leq 1$ such that $u_j(\hat{\mathbf{p}}) = y_j$, since $0 \leq y_j \leq x_j$. By construction $0 \leq \hat{p}_j^k \leq p_j^k$ and thus $\hat{\mathbf{p}} \leq \mathbf{p}$. $u_i(\cdot)$ is a non-increasing function of $\hat{p}_j^k$, which implies that $u_i(\hat{\mathbf{p}}) \geq u_i(\mathbf{p}) \geq y_i$, and thus $\mathbf{x}' = \mathbf{u}(\hat{\mathbf{p}}) \geq \mathbf{y}$.

Define a sequence as follows: $\mathbf{p}^0 = \mathbf{p}$ and $\mathbf{p}^n$ such that $\{p^n\}_i^k = \{p^{n-1}\}_i^k, \forall i \neq j$, where $j = (n \bmod N) + 1$ and $\{p^n\}_j^k = \alpha\{p^{n-1}\}_j^k$ such that $u_j(\mathbf{p}^n) = y_j$. Define $\mathbf{x}^n = \mathbf{u}(\mathbf{p}^n)$. We know that $\mathbf{x}^0 \geq \mathbf{y}$, so by the above argument, we know that $\mathbf{x}^1 \geq \mathbf{y}$ and that $\mathbf{p}^1 \leq \mathbf{p}^0$. By induction it follows that $\mathbf{p}^{n+1} \leq \mathbf{p}^n$.

The sequence $\mathbf{p}^n$ is monotonically non-increasing and is bounded below by $\mathbf{0}$. Therefore it must converge. Since $\mathbf{u}$ is a continuous function, the sequence $\mathbf{x}^n$ also converges, which implies that the sequence x_i^n, for all i, also converges. Given that every N^{th} element of the sequence x_i^n is equal to y_i, the sequence x_i^n must converge to y_i. Thus, $\mathbf{x}^n$ converges to $\mathbf{y}$. By the continuity of $\mathbf{u}(\cdot)$ and compactness of P^χ, $\mathbf{u}(\mathbf{p}^*) = \mathbf{y}$, where $\mathbf{p}^*$ is the limit of the sequence $\mathbf{p}^n$. Thus, $\mathbf{y} \in U$. ■

We have just shown that as the number of channels increases, the utility space for the spectrum sharing game approaches convexity. As the utility space approaches convexity, the NP maximizer of the utility space approximates the NP maximizer of the convex hull of the utility space, which is the unique NBS. By theorem 1, the NP maximizer for the utility space is also the unique NBS. Also, as shown in Fig. 2, we should expect the NBS to be approximately a PO extension of the KSS. In our simulation results, we will show that the NBS and the KSS achieve similar allocations, with the NBS providing the best compromise between efficiency and fairness.

V. Distributed Algorithm

Our goal is to design a distributed algorithm that achieves the NBS for the spectrum sharing game. We need the algorithm to operate only with local information and no centralized control. In this section we show that nodes can be aggregated into overlapping groups, which we can then leverage to distribute the computation of the NBS. Nodes within each group are in close proximity, which allows nodes to only use local information. Finally, we propose an algorithm for computing an approximation to the NBS and prove its convergence.

We make the following assumptions:

1) There is an underlying method for information exchange such that nodes within two hops of one another can communicate within a time scale shorter than the time scale for updates to channel allocation.
2) Nodes run the algorithm at random intervals such that the probability that two or more nodes (within two hops of each other) run the algorithm simultaneously is small.
3) The execution time of the algorithm is small relative to the interval between executions of the algorithm.

Before we discuss the algorithm, we need to introduce an approximation to the NP. The approximation transforms the NP into logarithmic form. It uses the following approximation of the log function, defined for all $x \in \mathbb{R}$:

$$l_\epsilon(x) = \begin{cases} \log(\epsilon) + \frac{1}{\epsilon}(x-\epsilon) & \text{if } x < \epsilon, \\ \log(x) & \text{otherwise.} \end{cases} \quad (12)$$

where $\epsilon > 0$. For a small enough value of ϵ, equation (1) is equivalent to the following expression:

$$\arg\max_{\mathbf{u}\in U} \sum_{i=1}^{N} l_\epsilon(u_i - u_i^0). \quad (13)$$

Proposition 1: Equation (1) is equivalent to equation (13).

Proof: We know that there exists $\mathbf{u} > \mathbf{u}^0$, which implies the existence of $\epsilon_1 > 0$ such that $u_i - u_i^0 > \epsilon_1$. Let

$$a = (N-1)\log\left(\max_i \left\{h_i(U, \mathbf{u}^0)\right\}\right), \quad (14)$$

$b = N\log(\epsilon_1)$ and $0 < \epsilon < \min(\epsilon_1, e^{(b-a)})$.

We will show that (13) is equal to (1). First consider the case where $\mathbf{u}^1 \in U$ such that $u_i^1 > u_i^0 + \epsilon$, for all i. Then, $l_\epsilon(u_i^1 - u_i^0) = \log(u_i^1 - u_i^0)$, and the result follows. Now consider the case where $\mathbf{u}^2 \in U$ such that there exists an i such that $u_i^2 < u_i^0 + \epsilon$. Then,

$$\sum_j l_\epsilon(u_j^2 - u_j^0) < \log(\epsilon) + a = b < \sum_j \log(u_j^1 - u_j^0). \quad (15)$$

Therefore, $\mathbf{u}^2$ is never the maximizer of (13). Thus, for $\mathbf{u}$ to be a maximizer of (13), it must satisfy $u_i - u_i^0 > \epsilon$ which implies $l_\epsilon(u_i - u_i^0) = \log(u_i - u_i^0)$, for all i. ■

Notice that the expression in (13) now maximizes over the entire set U. This is crucial for the algorithm not to have to account for scenarios where a node currently achieves a utility lower than the disagreement point, which may happen in intermediate steps before the algorithm converges. For the rest of this document, when we refer to maximizing the NP, we refer to (13).

The NBS solution is based on the assumption that all players in the game bargain as a group to reach a cooperative solution to the game. This means that in the spectrum sharing game a node cooperates with all nodes in the network. That is, we must consider the utility achieved by all nodes in the network in order to implement the NBS. However, we know that a node's effect is limited to the other nodes within close proximity. This allows us to limit the scope of a node's bargaining to a subset of the network. Consider the following concept:

Definition 5: Let $R > 0$ and $Rx(i)$ be the set of receiver nodes of node i. The *interference zone* (IZ) of node j, with interference radius R, is defined as:

$$IZ_R(j) = \{i | \exists k \in Rx(i), \texttt{distance}(j,k) < R, j \neq k\}. \quad (16)$$

The interference zone for node j is the set of transmitter nodes such that one of their receivers is within distance R of node j. If we set R to a large enough value, then node $i \notin IZ_R(j)$ can ignore node j's actions. Thus, we can approximate the utility function of node i as follows:

$$\tilde{u}_i(\mathbf{p}) = \frac{B}{K}\sum_{k=1}^{K} \log_2\left(1 + \frac{H_{ii}^k p_i^k}{\frac{\sigma^2}{K} + \sum_{l\in J_i} H_{li}^k p_l^k}\right) \quad (17)$$

where $J_i = \{j | i \in IZ_R(j)\}$. This approximation drops the interference terms from nodes that are far enough away from node i's receivers such that they cause negligible interference. Then, the utility function of node $i \notin IZ_R(j)$ is independent of node j's actions.

Consider node j maximizing the NP while the other nodes' actions remain constant. Let $\mathbf{p}' \in P^\chi$ be the current strategy employed in the network, $\rho_j = \{\mathbf{p} | \mathbf{p} \in P^\chi, p_i^k = {p'}_i^k, \forall i \neq j\}$ be the set of strategies such that only node j has an action different from $\mathbf{p}'$, and $IZ_R^+(j) = IZ_R(j) \cup \{j\}$. Then,

$$\max_{\mathbf{p}\in\rho_j} \sum_i l_\epsilon(\tilde{u}_i(\mathbf{p}) - u_i^0) = \sum_{l\notin IZ_R(j)} l_\epsilon(\tilde{u}_l(\mathbf{p}') - u_l^0) * \max_{\mathbf{p}\in\rho_j} \sum_{i\in IZ_R^+(j)} l_\epsilon(\tilde{u}_i(\mathbf{p}) - u_i^0). \quad (18)$$

Equation (18) demonstrates that, by using the approximation for the utility function, nodes need only consider nodes in their IZ when maximizing the NP. This result allows us to design an algorithm to calculate the NP for the entire network only using local information at each node.

We propose Algorithm 1, which consists of nodes choosing their actions so as to maximize the NP of their interference zone. The algorithm only updates the node's actions if the value of the NP is increased by at least tol percent. The

Algorithm 1 Distributed NBS Computation

$IZ = i \cup \{j|\mathtt{distance}(Rx(j), i) < R, u_j > 0\}$
$oldNP = \sum_{j \in IZ} l_\epsilon(u_j(\mathbf{p}) - u_j^0)$
$\hat{\mathbf{p}} = \mathtt{MaximizeNP}(i, IZ, \delta)$
$newNP = \sum_{j \in IZ} l_\epsilon(u_j(\hat{\mathbf{p}}) - u_j^0)$
if $newNP > (1 + tol) * oldNP$ **then**
  $p_i^k = \hat{p}_i^k$
end if

algorithm calls the function $\mathtt{MaximizeNP}(i, IZ, \delta)$, which calculates the maximum of the NP for all nodes in the IZ with respect to node i's actions, with the following condition:

$$u_i(\hat{\mathbf{p}}) \leq u_i(\mathbf{p}) + \delta * h_i(U, \mathbf{0}) \tag{19}$$

where $\delta > 0$, $\mathbf{p}$ is the action set before execution of the algorithm and $\hat{\mathbf{p}}$ is the action set after execution of the algorithm. The following proof shows the convergence of the algorithm. Evaluating the efficiency of the convergent states analytically is a non-trivial problem. We carry out an experimental analysis to show that convergent points closely approximate the NBS.

Proof of correctness of Algorithm 1 and convergence: Consider a non-cooperative game where the utility of player j is:

$$\sum_{i \in IZ_R^+(j)} l_\epsilon(\tilde{u}_i(\mathbf{p}) - u_i^0). \tag{20}$$

It can be shown that this game is a potential game [7] with the following as the potential function:

$$\sum_i l_\epsilon(\tilde{u}_i(\mathbf{p}) - u_i^0). \tag{21}$$

A known result of potential games is that when the utility of all players is upper-bounded and all players act selfishly (their action will never result in a decrease of utility), the game converges to a Nash Equilibrium (NE) [8].

If a node executes the algorithm, the potential function of the non-cooperative game does not decrease. Therefore, the node is acting selfishly and the algorithm will converge. ∎

VI. Simulation Results

In this section we present our simulation results. All simulations adopt the following setup. There are N transmitter-receiver pairs of nodes placed randomly in an R_a meter by R_a meter square. A receiver is no more than 100m away from its transmitter. The total bandwidth, B, is evenly divided into K channels. The propagation loss exponent is 4 and the root-mean-square (RMS) delay spread is $1\mu s$. The antenna gain is 0.01, the maximum transmission power is 100mW, and the noise level is -80dBm for the entire bandwidth. All capacity numbers are expressed in bits/sec per Hz and are the actual capacity achieved, not the approximation used by the algorithm. Unless otherwise stated, all simulations are averaged over 100 randomly chosen network topologies, $N = 10$, $K = 10$, $tol = 2\%$ and disagreement point of 0 for all nodes.

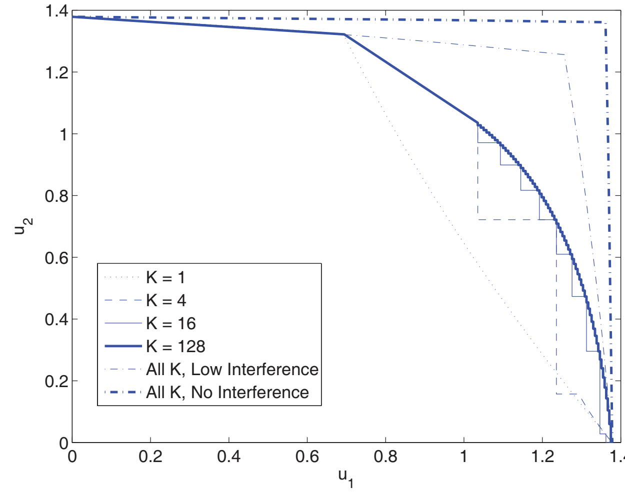


Fig. 4. WPO set for 2 player game under mixed interference for various values of K.

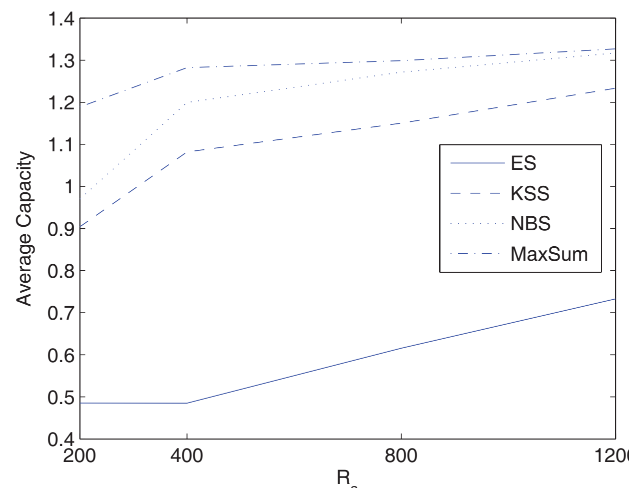


Fig. 5. Average capacity achieved by the various schemes.

A. Bargaining Solution Comparison

In this section we investigate the spectrum allocation achieved by the bargaining solutions. We compare them using the following metrics: the average, minimum and standard deviation of capacity, the KSS score and NBS score. The KSS score is defined as:

$$\min_i \{\frac{1}{\theta_i}(u_i - u_i^0)\}$$

and the NBS score is defined as:

$$\sqrt[N]{\prod_i \frac{u_i - u_i^0}{u_i^* - u_i^0}}$$

where $\mathbf{u}^*$ is the NBS allocation. We also compare the bargaining solutions to the allocation that maximizes the sum of the capacities (MaxSum). In [1], we compared performance of the NBS to that of the non-cooperative game (water-filling). Here, we focus on comparing the different bargaining solutions with respect to the fairness and efficiency achieved.

To compare the spectrum allocation achieved by the various bargaining solutions, we wish to examine how each balances efficiency and fairness. The average capacity captures the

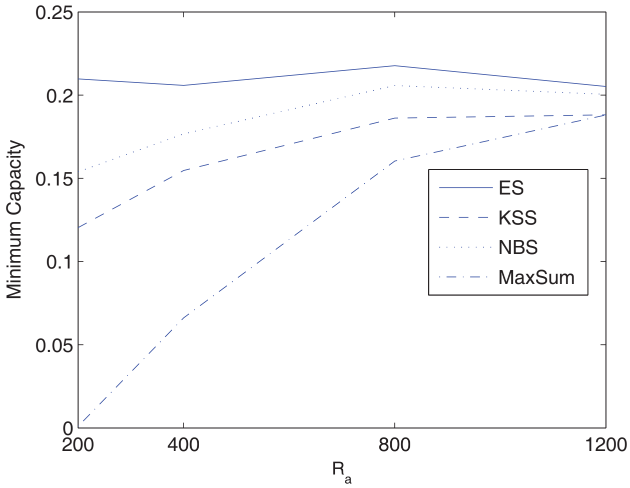


Fig. 6. Minimum capacity achieved by the various schemes.

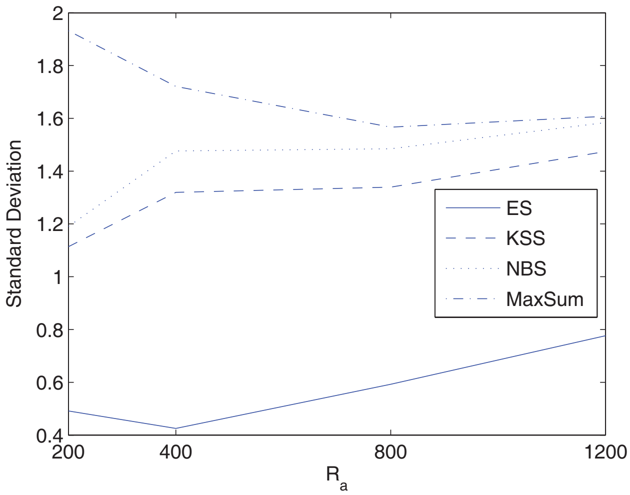


Fig. 7. Standard deviation of capacity achieved by the various schemes.

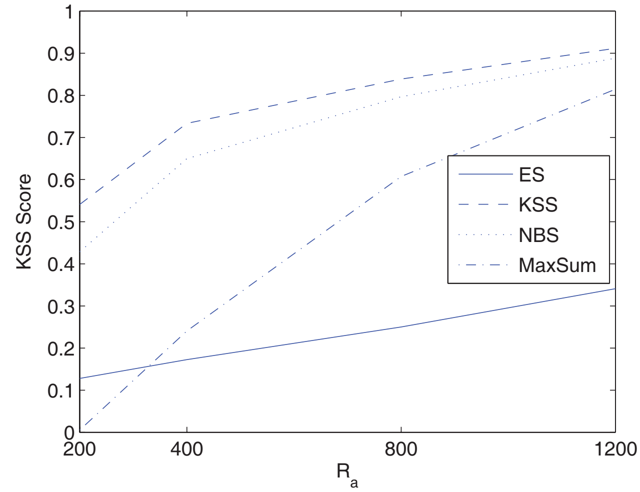


Fig. 8. KSS Score achieved by the various schemes.

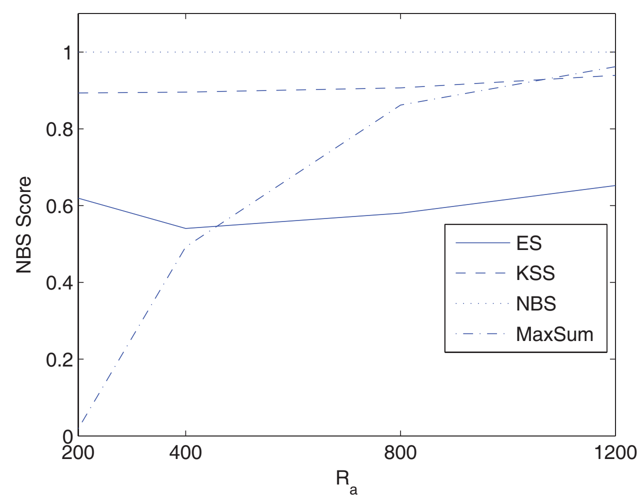


Fig. 9. NBS Score achieved by the various schemes.

efficiency of the allocation. The other four metrics capture the fairness of the allocation. The ES aims to maximize the minimum capacity achieved. The KSS and NBS scores are the values that the KSS and the NBS aim to maximize, respectively. We also show the standard deviation to quantify the variability of the capacities achieved. It is desirable to decrease variability when considering fairness.

Figure 5 shows the average capacity for different values of R_a. We expect, by definition, that MaxSum achieves the highest average capacity. The ES achieves the lowest average capacity, as it tries to maximize the minimum, not the average. Also, average capacity should not decrease with R_a, as we expect less interference as R_a increases. All these effects can be readily observed from the figure. Additionally, we observe that the NBS achieves a higher capacity than KSS. More importantly, the NBS catches up to MaxSum faster than the KSS as R_a increases (density decreases).

Figure 6 shows the minimum capacity achieved. The ES, by definition, achieves the highest minimum capacity and as expected, MaxSum achieves the lowest. In fact, we can observe than for $R_a = 0$, the minimum achieved by MaxSum is near 0. Similar to the average capacity, the NBS outperforms the KSS, and it also catches up to the ES faster than the KSS. Again, as R_a increases (and density decreases), all solutions achieve similar values for the minimum capacity.

Figure 7 shows the standard deviation of capacity achieved. By definition, in the ES all nodes achieve the same capacity. However, we examine the PO-extension of the ES which permits nodes to achieve capacity higher than the minimum. As expected the ES achieves the lowest and MaxSum achieves the highest standard deviation of capacity. The NBS and KSS achieve a standard deviation of capacity significantly higher than the ES, with the KSS being slightly lower than the NBS.

Figure 8 and Figure 9 show the KSS score and the NBS score achieved, respectively. By definition, the KSS achieves the highest KSS score and the NBS achieves the highest NBS score. We can observe that for small values of R_a, the ES and MaxSum achieve poor KSS and NBS scores. When density is high, there is significant interference and the minimum achieved by both ES and MaxSum is small. This leads to equally small KSS and NBS scores. As R_a increases, MaxSum performs similarly to, but still worse than, the KSS and NBS on both scores. This is expected, as interference diminishes and there is no longer the need to balance efficiency

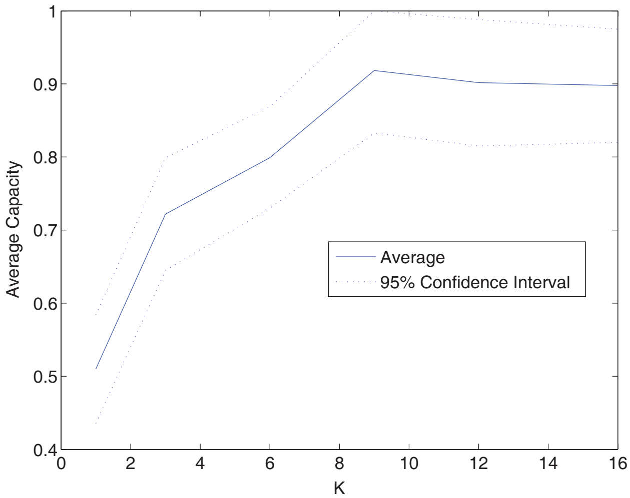


Fig. 10. Average capacity per node achieved by the NBS ($R_a = 200m$).

with fairness. Both the NBS and KSS slightly outperform each other on their respective scores, with both significantly outperforming ES and MaxSum.

From these results we can reach the following conclusions. The ES is best at minimizing the deviation between the capacity achieved by the nodes but achieves a poor average capacity. The MaxSum, as expected, achieves the best average capacity, but performs poorly on all other metrics. The KSS and the NBS achieve good average capacity while also performing well in the other metrics. However, the NBS edges over the KSS in average capacity as density decreases and achieves a higher minimum capacity. For these reasons, we choose to implement the NBS in our algorithm for fair and efficient spectrum sharing.

B. Utility Space Efficiency

Figure 10 shows the average capacity per node achieved by the NBS, for several values of K, with $R_a = 200m$. Since R_a is small relative to the transmission range, we expect significant contention for the spectrum. As we showed in the previous section, we see that under a high interference environment, the average capacity per node increases as the number of channels increases. We see that after a certain point ($K = 9$), the average capacity stops increasing. This is not surprising, as K is close to N, which is the point where every node can utilize a single channel exclusively and thus avoid interference. Beyond the point where $K = N$, there are no further gains as every node is already able to transmit without interference and the amount of spectrum available to each node remains constant.

C. Algorithm Performance

Figure 11 examines the effect of the parameter δ (for $R_a = 200$), which controls the amount of the space the algorithm can search at a given iteration. We can see that as $\delta \rightarrow 1$ the algorithm performs rather poorly. This is because the search space is unconstrained and the first node to execute the algorithm will skew the allocation in its favor. By limiting the space each node can explore at any iteration, it limits the skewing of the allocation. The figure shows how making δ smaller, the algorithm converges to a higher fairness score. However, this comes at a cost of slower convergence. The figure shows the average number of round robin iterations required for the algorithm to converge. It clearly shows that the algorithm requires more iterations to converge as δ decreases. The results in Figure 11 are for a high interference environment. We expect the performance of the algorithm to improve (better fairness score, fewer iterations to converge) as interference is reduced. Thus, these results represent the expected worst case performance of the algorithm. For the rest of the simulations we set parameter $\delta = 0.2$, as that is the point where we achieve the best compromise between fairness score and convergence speed, as Figure 11 illustrates. In practice, the choice of optimal δ is a design decision that depends on the marginal benefit of achieving improved fairness in the system and the marginal cost of increased convergence time.

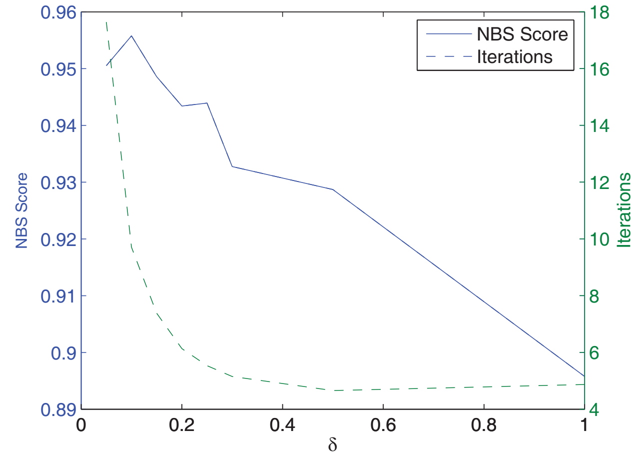


Fig. 11. Effect of delta on average score ($K = 10$, $R_a = 200m$).

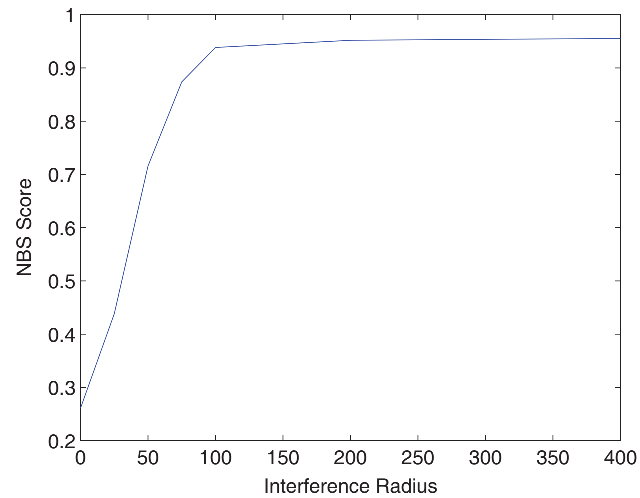


Fig. 12. Performance of the algorithm versus Interference Radius (R); $K = 20$, $N = 20$ and $R_a = 600$.

In Figure 12, we show the performance of the algorithm as a function of the interference radius. Clearly, we would like this radius to be as small as possible, so as to minimize the information exchange required. As expected, the fairness score increases as R increases. However, we notice that even when $R = 100$, the fairness score achieved by the algorithm is greater than 0.94. This result is encouraging as it tells us that

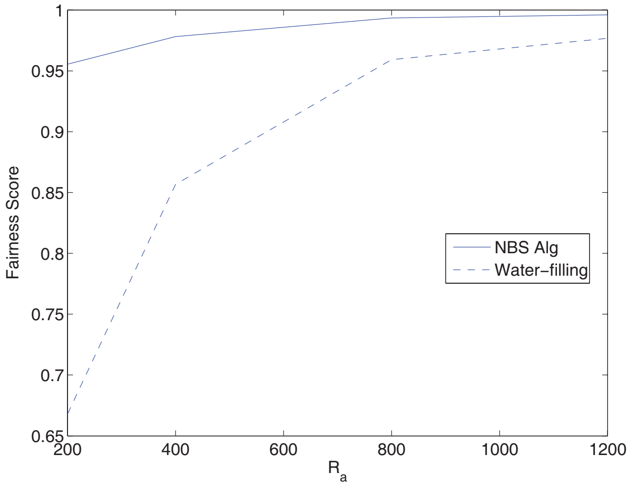


Fig. 13. Performance for various values of R_a.

we can significantly limit the number of nodes involved in the bargaining process and still achieve a reasonable outcome.

Finally, we want to examine the fairness score achieved by the algorithm relative to the non-cooperative water-filling scheme. Figure 13 shows the fairness score achieved by both algorithms under different values of R_a. As R_a increases, the nodes are spaced farther apart and thus experience less interference from the other nodes. The need to cooperate diminishes, the less interference nodes can cause to each other. This is confirmed by the results in Figure 13, which shows that the water-filling algorithm achieves significantly lower fairness scores than the NBS for small values of R_a. As R_a increases, the gap between the NBS and the result of water-filling becomes smaller.

VII. Related Work

We discuss a few notable studies related to game theory, capacity maximization, and spectrum sharing in wireless networks, and then position the contributions of our work.

In the literature on wireless networks, the use of *non-cooperative game theory* is perhaps more mature than its co-operative counterpart. Non-cooperative game theory is utilized in [9] for power control problems, where a pricing technique is used to achieve the best Pareto improvement of user throughput over the initial non-cooperative NE. In [10], a non-cooperative game approach for distributed resource allocation is employed. Two game models are proposed: a power-control game at the user level, and a throughput game at the system level. A referee-based non-cooperative game is proposed in [11] and [12] to perform sub-channel assignment, adaptive modulation, and power control for multi-cell multi-user orthogonal frequency division multiple access (OFDMA) networks. In [13] and [14], interference avoidance (IA) schemes are formulated as a potential game for a distributed waveform adaptation mechanism by which multi-access interference can be reduced in a wireless communication system. In [15], a channel selection scheme is formulated as a potential game. The authors formalize a low complexity distributed ad-hoc dynamic frequency selection algorithm that converges to near-minimal interference frequency re-use patterns. In [16], a game formulation using the theoretical capacity of a channel as the utility function is proposed. The authors show that for the game where players are capable of spreading power over the complete spectrum with arbitrary precision (analogous to an infinite number of channels), all achievable utility points can be obtained using only pure strategies, with piece-wise constant power allocations. In [17], non-cooperative power control is used to analyze the outage probability in multicell data networks. In [18], iterative water-filling is used for resource allocation in multiple-access Gaussian vector channel. Our work is similar to all these, in that we too consider the problem of optimal power allocation, with users maximizing their individual link capacities. Unlike all these studies, which only consider system efficiency, we examine the fundamental question of how to allocate resources (power) in a fair and efficient way. To a large extent, non-cooperative game-theoretic models can handle the issue of how to efficiently allocate network resources (this is particulary true in potential games [7], [8], [13]–[15]). These models, however, are somewhat limited when addressing fairness in conjunction with efficiency. To cope with these two issues, *cooperative game theory* is often used.

A cooperative game-theoretic framework is described in [19] for bandwidth allocation for elastic services in high-speed networks. In [20], the Nash Bargaining Solution is studied for power, rate, and subcarrier allocation for single-cell OFDMA systems to achieve fair and efficient performance. In [21], a bargaining approach is proposed, where a *poverty line* is used to indicate the minimum spectrum usage for a node. Our work differs from [19] in that we study the wireless medium. The non-convexity of the optimization problem we consider prevents us from deriving analytical solutions similar to those in [19]–[21]. Unlike that in [20], our framework does not assume that receivers are necessarily co-located. Since receivers are not co-located, a centralized solution is not feasible.

Other variants of non-cooperative game theory approach to analyze wireless networks have also been proposed. In [22], the authors use Markov chains to model the multiple access problem as a repeated game with perfect information. *Repeated game theory* studies the behaviors of users at multiple stages to dynamically optimize the wireless resource allocation over time. Using repeated games, the authors in [16] argue that the possibility of building reputations and applying punishments allows a larger set of achievable rates, which can then support a NE that is efficient. An *auction theory* based approach is proposed in [23] for sharing spectrum among a group of users, subject to a constraint on the interference temperature at co-located receivers. In the context of cognitive radio networks, a spectrum trading game modeling primary and secondary user behavior is analyzed in [24]. A pricing approach is used in [25] to study the spectrum sharing problem where primary users offer spectrum access opportunities to secondary users and maximize profits subject to QoS constraints. Here, the secondary user behavior is modeled as a game; collusion and repeated game models are utilized to achieve equilibria that are efficient (i.e. maximize the sum of primary user profits). Some methods in synthesis with *graph theory* have been also proposed. In [26], the authors study the channel assignment problem as a game. They formulate the game as a maximal coloring graph problem and find the NE of the game by solving the coloring problem. We formulate a game similar to the one studied in [16]; unlike [16] and other non-cooperative models, we take a cooperative game approach, and develop a distributed scheme that aims to achieve the operating point dictated by the NBS. It has been noted in [27] that cooperative spectrum sharing systems achieve many of the same long term benefits in terms of increasing spectrum utilization (probably without raising as much opposition from traditional spectrum users) as the non-cooperative systems.

Fairness is an important consideration in the distributed allocation of spectrum. Different fairness definitions have been proposed in the literature, including proportional fairness [28], max-min fairness [29], long term time average fairness [30], etc. In [31], it is shown that optimization formulations based on product forms can lead to optimality and fairness. The NBS, which is based on the product of individual user performance objectives, is known to be a suitable candidate for a fair, optimal operating point.

VIII. Conclusions

The spectrum sharing problem consists of dividing a given amount of spectrum among many nodes in a way that is efficient and fair. In this manuscript we have addressed this problem by formulating a cooperative game model of the spectrum sharing problem. We analyzed the utility space of the spectrum sharing game and showed that efficiency is maximized by increasing the number of channels. We also showed that the utility space convexifies and thus maximizing the NP gives us an allocation that approximately satisfies the NBS axioms. Consequently, we showed that the NBS allocation provides a reasonable compromise between efficiency and fairness, as it achieves allocations with minima close to the ES and efficiency close to the MaxSum. Finally, we proposed an algorithm that, with only local information, approximates the maximization of the NP and showed that it converges quickly to a value close to the NBS.

References

[1] J. E. Suris, L. DaSilva, A. MacKenzie, and Z. Han, "Cooperative game theory for distributed spectrum sharing," in *Proc. IEEE Int'l Conference on Communcations (ICC 2007)*, pp. 5282–5287, June 2007.

[2] J. Nash, "Two-person cooperative games," *Econometrica*, vol. 21, pp. 128–140, Jan. 1953.

[3] H. J. M. Peters, *Axiomatic Bargaining Game Theory*. Kluwer Academic Publishers, 1992.

[4] E. Kalai and M. Smorodinsky, "Other solutions to Nash's bargaining problem," *Econometrica*, vol. 43, pp. 513–518, May 1975.

[5] L. Zhou, "The Nash bargaining theory with non-convex problems," *Econometrica*, vol. 65, pp. 681–685, May 1997.

[6] J. Conley and S. Wilkie, "The bargaining problem without convexity—extending the egalitarian and Kalai-Smorodinsky solutions," *Economic Lett.*, vol. 36, pp. 365–369, Aug. 1991.

[7] D. Monderer and L. Shapley, "Potential games," *Games and Economic Behavior*, vol. 14, pp. 124–143, 1996.

[8] J. W. Friedman and C. Mezzetti, "Learning in games by random sampling," *J. Economic Theory*, vol. 98, pp. 55–84, May 2001.

[9] C. Saraydar, N. Mandayam, and D. Goodman, "Efficient power control via pricing in wireless data networks," *IEEE Trans. Commun.*, vol. 50, pp. 291–303, Feb. 2002.

[10] Z. Han and K. J. R. Liu, "Noncooperative power-control game and throughput game over wireless networks," *IEEE Trans. Commun.*, vol. 53, pp. 1625–1629, Oct. 2005.

[11] Z. Han, Z. Ji, and K. J. R. Liu, "Non-cooperative resource competition game by virtual referee in multi-cell OFDMA networks," *J. Select. Areas Commun.*, Special Issue on Non-cooperative Behavior in Networking, vol. 53, pp. 1079–1090, Aug. 2007.

[12] Z. Han and K. J. R. Liu, *Resource Allocation for Wireless Networks: Basics, Techniques, and Applications*. Cambridge University Press, Apr. 2008.

[13] J. E. Hicks, A. B. MacKenzie, J. O. Neel, and J. H. Reed, "A game theory perspective on interference avoidance," in *Proc. IEEE Global Communications Conference (GLOBECOM 2004)*, pp. 257–261, Dec. 2004.

[14] R. Menon, A. MacKenzie, R. Buehrer, and J. Reed, "Game theory and interference avoidance in decentralized networks," in *SDR Forum Technical Conference*, Nov. 2004.

[15] J. Neel and J. Reed, "Performance of distributed dynamic frequency selection schemes for interference reducing networks," in *Proc. Military Communications Conference (MILCOM 2006)*, pp. 1–7, 2006.

[16] R. Etkin, A. Parekh, and D. Tse, "Spectrum sharing for unlicensed bands," in *Proc. IEEE Int'l Symp. on New Frontiers in Dynamic Spectrum Access Networks (DySPAN 2005)*, pp. 251–258, 2005.

[17] L. He and J. Walrand, "Pricing and revenue sharing strategies for internet service providers," *IEEE J. Select. Areas Commun.*, vol. 24, pp. 942–951, May 2006.

[18] W. Yu, W. Rhee, S. Boyd, and J. Cioffi, "Iterative water-filling for gaussian vector multiple-access channels," *IEEE Trans. Inform. Theory*, vol. 50, no. 1, pp. 145–152, 2004.

[19] A. Jalali, R. Padovani, and R. Pankaj, "Data throughput of CDMA-HDR—a high efficiency-high data rate personal communication wireless system," in *Proc. Vehicular Technology Conference*, vol. 3, pp. 1854–1858, 2000.

[20] Z. Han, Z. Ji, and K. J. R. Liu, "Fair multiuser channel allocation for OFDMA networks using Nash bargaining solutions and coalitions," *IEEE Trans. Commun.*, vol. 53, pp. 1366–1376, Aug. 2005.

[21] L. Cao and H. Zheng, "Distributed spectrum allocation via local bargaining," in *Proc. IEEE Communications Society Conference on Sensor and Ad Hoc Communications and Networks (SECON 2005)*, pp. 475–486, 2005.

[22] A. B. MacKenzie and S. B. Wicker, "Stability of multipacket slotted aloha with selfish users and perfect information," in *Proc. IEEE INFOCOM 2003*, pp. 1583–1590, 2003.

[23] J. Huang, R. Berry, and M. Honig, "Auction-based spectrum sharing," *ACM/Kluwer J. Mobile Networks and Applications (MONET)*, vol. 11, pp. 405–418, June 2006.

[24] D. Niyato, E. Hossain, and Z. Han, "Dynamics of multiple-seller and multiple-buyer spectrum trading in cognitive radio networks: a game theoretic modeling approach," *IEEE Trans. Mobile Computing*, vol. 8, pp. 1009–1022, Aug. 2009.

[25] D. Niyato and E. Hossain, "Competitive pricing for spectrum sharing in cognitive radio networks: dynamic game, inefficiency of Nash equilibrium, and collusion," *IEEE J. Select. Areas Commun.*, vol. 26, pp. 192–202, Jan. 2008.

[26] M. H. Magnus, Y. Joseph, L. Li, and V. Mirrokni, "On spectrum sharing games," in *Proc. ACM Symp. on Principles of Distributed Computing*, pp. 107–114, 2004.

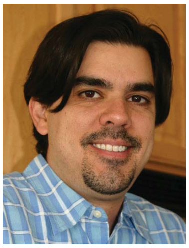
[27] M. J. Marcus, "CR: cooperative radio or confrontational radio," in *Proc. 2nd IEEE Intl Symp. on New Frontiers in Dynamic Spectrum Access Networks, Networks (DySPAN 2007)*, pp. 208–211, 2007.

[28] F. P. Kelly, A. Maulloo, and D. Tan, "Rate control in communication networks: shadow prices, proportional fairness and stability," *J. Operational Research Society*, vol. 49, pp. 237–252, 1998.

[29] L. Tassiulas and S. Sarkar, "Maxmin fair scheduling in wireless networks," in *Proc. IEEE INFOCOM 2002*, vol. 2, pp. 763–772, 2002.

[30] Z. Han and K. J. R. Liu, "Power minimization under throughput management over wireless networks with antenna diversity," *IEEE Trans. Wireless Commun.*, vol. 3, pp. 2170–2181, Nov. 2004.

[31] R. Mazumdar, L. Mason, and C. Douligeris, "Fairness in network optimal flow control: optimality of product," *IEEE Trans. Commun.*, vol. 39, pp. 775–782, May 1991.



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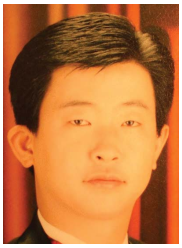

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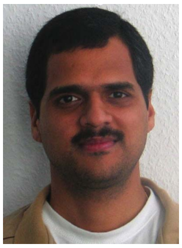

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